# Grade 7/8 Math Circles

## Nov 7/8/9/10, 2022

## Induction - Problem Set

1. True or False? Provide an explanation.
   (i) For all $n \in \mathbb{N}$, $n < n^2$.
   (ii) For all $n \in \mathbb{N}$, $n + 2 \leq 3^n$.
   (iii) For all $n \in \mathbb{N}$, $\frac{1}{3}(n+1)(n+2) \in \mathbb{N}$.
   (iv) For all $n \in \mathbb{N}$, $n^2 - 1 = (n-1)(n+1)$.

2. For $n \in \mathbb{N}$, let $F_n$ be the $n$th Fibonacci number. That is, let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that
$$F_{n+2} = 1 + F_1 + F_2 + ... + F_n$$
for all $n \in \mathbb{N}$.

3. Use induction to prove the Pigeonhole Principle: 'For all $n \in \mathbb{N}$, if $n+1$ pigeons occupy $n$ holes, then some hole must have at least 2 pigeons.'

4. The following proof shows that every horse has the same colour. Find the mistake:
Let $P(n)$ be the statement: in any group of $n$ horses, all horses have the same colour.
   - Base Case: If there is only one horse in a group, then there is only one colour of horse in that group. Therefore $P(1)$ is true.
   - Inductive Hypothesis: Let $k$ be a natural number and assume $P(k)$ is true. That is, in any group of $k$ horses, all horses have the same colour.
   - Inductive Step: Consider any group of $k + 1$ horses. Remove a horse from the group. By the inductive hypothesis, all other horses must have the same colour. Add the horse back to the group and remove a different horse. Once again, all horses have the same colour. Therefore, the horse we initially removed has the same colour as the other $k$ horses. We can conclude that all horses in this group have the same colour.
   - Conclusion: By the Principle of Mathematical Induction, for all $k \in \mathbb{N}$, any group of $k$ horses has the same colour.

   Since there are a finite number of horses in the world, all horses have the same colour.

5. Prove that $1 + \frac{n}{2} \leq 1.5^n$ for all $n \in \mathbb{N}$.

6. For all natural numbers $n \geq 3$, prove that the sum of the interior angles of a polygon with $n$ sides is $180° \times (n - 2)$.

   *Hint: create a triangle in your inductive step*

7. Prove that any natural number can be written as the sum of (one or more) distinct powers of 2 (the powers of 2 are $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, ...). For example, $19 = 16 + 2 + 1 = 2^4 + 2^1 + 2^0$.

   *Hint 1: use strong induction*

   *Hint 2: when trying to write $n \in \mathbb{N}$ as a sum of distinct powers of 2, consider the largest power of 2 which is at most $n$.*

8. Bonus Problem: A group of 1000 people lives on a mysterious island. Every person has either blue eyes or green eyers. None of them know their eye colour, and they are forbidden to discuss the topic; thus, each resident can (and does) see the eye colours of all other residents, but has no way of discovering their own (there are no reflective surfaces). If someone discovers that they have blue eyes, then they must leave the island on the following day, in front of the entire the village. Assume every person on the island is highly logical, and knows that everyone else is highly logical.

   Of the 1000 islanders, it turns out that 100 of them have blue eyes and the rest have green eyes, although the islanders are not initially aware of these statistics (each of them can of course only see the other 999 people). One day, a foreigner visits to the island, winning the complete trust of the islanders. Not knowing the customs, the foreigner makes the mistake of mentioning eye colour in their address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

   What effect, if anything, do the foreigner's words have on the residents of the island?

   *Hint: What would happen if only 1 islander had blue eyes? What if only 2 islanders had blue eyes?*

   (Adapted from a post on Terence Tao's blog)